ON IRREDUCIBLE SYMPLECTIC 4-FOLDS NUMERICALLY EQUIVALENT TO $(K3)^{[2]}$

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ABSTRACT. We study the conjecture of O'Grady about irreducible symplectic 4-fold numerically equivalent to the Douady space $(K3)^{[2]}$.

1. INTRODUCTION

A Kähler manifold X is irreducible symplectic if it is simply connected and has a holomorphic symplectic form spanning $H^0(\Omega_X^2)$. Recall that two irreducible symplectic manifolds M_1, M_2 of dimension $2n$ are numerically equivalent if there exists an isomorphism of abelian groups

$$\psi \colon H^2(M_1, \mathbb{Z}) \to H^2(M_2, \mathbb{Z})$$

such that $\int_{M_1} \alpha^{2n} = \int_{M_2} \psi(\alpha)^{2n}$ for all $\alpha \in H^2(M_1, \mathbb{Z})$. The aim of this paper is to continue the O'Grady program of classification of projective irreducible symplectic manifolds, by proving in some cases the O'Grady conjecture [O, Conj. 1.2] (explained below). The following theorem is proved in [O, Prop. 3.2, Prop. 4.1]:

Theorem 1.1 (O'Grady). *Let M be a symplectic 4-fold numerically equivalent to $(K3)^{[2]}$. There exists an irreducible symplectic manifold X deformation equivalent to M such that:*

(1) *X has an ample divisor H with $(h, h) = 2$ (i.e. $H^2 = 12$), where $h := c_1(H)$,*
(2) *$H_{\mathbb{Z}}^{1,1}(X) = \mathbb{Z}h$,*
(3) *if $\Sigma \in Z_1(X)$ is an integral algebraic 1-cycle on X and $cl(\Sigma) \in H_{\mathbb{Q}}(X)$ is its Poincaré dual, then $cl(\Sigma) = mh^3/6$ for some $m \in \mathbb{Z}$,*
(4) *if $H_1, H_2 \in |H|$ are distinct then $H_1 \cap H_2$ is a reduced irreducible surface,*
(5) *if $H_1, H_2, H_3 \in |H|$ are linearly independent, the subscheme $H_1 \cap H_2 \cap H_3$ has pure dimension 1 and the Poincaré dual of the fundamental cycle $[H_1 \cap H_2 \cap H_3]$ is equal to h^3,*
(6) *$\chi(\mathcal{O}_X(nH)) = \frac{1}{2}n^4 + \frac{5}{2}n^2 + 3$, $n \in \mathbb{Z}$.*

Let us fix X and $h := c_1(H)$ as above. By Theorem 1.1(6) we have $\dim |H| = 5$. O'Grady conjectured that the map given by $|H|$ is not birational. This would imply by the results of [O, Thm. 1.1], that an irreducible symplectic 4-fold numerically equivalent to $(K3)^{[2]}$ is deformation equivalent to a natural double cover of an Eisenbud-Popescu-Walter sextic (see [O1]). Moreover, O'Grady proved that a natural double cover of such a generic sextic is deformation equivalent to a $(K3)^{[2]}$. Thus the conjecture imply that an irreducible symplectic 4-fold numerically equivalent to $(K3)^{[2]}$ is a deformation of $(K3)^{[2]}$.

So suppose that $\varphi_{|H|} : X \dashrightarrow X' \subset \mathbb{P}^5$ is birational. The hypersurface X' is non-normal unless $d = 6$. From [O, (4.0.25)], $X' \subset \mathbb{P}^5$ is a hypersurface of degree $6 \leq d \leq 12$. Our goal is to prove the conjecture in the case $8 \geq d \geq 6$, more precisely we obtain the following.

Theorem 1.2. *If the linear system $|H|$ on X defines a birational map $\varphi : X \dashrightarrow \mathbb{P}^5$ onto its image then $|H|$ has 0-dimensional base locus of length ≤ 3.*

If $6 < d < 12$ the fourfold X cannot be the normalization of X'. That is why we choose a generic codimension 2 linear section X'_D of $X' \subset \mathbb{P}^5$ that avoids the image of the curves contracted by φ. Denote by D the pre-image of X'_D on X. We construct the normalization Y_D of X'_D by a sequence of blow-ups an blow-downs of D. The surface Y_D has rational singularities so that we can find the possible cohomology tables of the conductor ideal of this normalization. In the case $6 < d \leq 8$ such ideal cannot exist. In the case $9 \leq d \leq 11$ we reduce the problem of existence of such idela to a problem of existence of an appropriated module of finite length over the polynomial ring.

If $d = 12$ we show that the subscheme $C \subset \mathbb{P}^5$ defined by the conductor of the normalization of X' is arithmetically Buchsbaum. We compute, using the properties of X, the degree, the number of minimal generators of $C \subset \mathbb{P}^5$, and the minimal resolution of the ideal $\mathcal{I}_C$ (it is uniquely determined). A subscheme with such invariants exist (even a smooth one) we can try to find a counterexample to the conjecture by considering a normalization of a degree 12 hypersurface singular along C. This case needs different methods and will be treated in a future paper.

2. Preliminaries

Let $\beta : Y \to X' \subset \mathbb{P}^5$ be the normalization of a hypersurface in $\mathbb{P}^5$. Denote by Z the subscheme of $\mathbb{P}^5$ defined by the adjoint ideal $\mathrm{adj}(X') \subset \mathcal{O}_{\mathbb{P}^5}$ (see [L, Def. 9.3.47]). From [L, Prop. 9.3.48] one has the following exact sequence:

$$(2.1) \qquad 0 \to \mathcal{O}_{\mathbb{P}^5}(-6) \to \mathcal{O}_{\mathbb{P}^5}(d-6) \otimes \mathcal{I}_Z \to \beta_*(\mathcal{O}_Y(K_Y)) \to 0.$$

Set $\mathcal{C} := \mathrm{Ann}(\beta_* \mathcal{O}_Y / \mathcal{O}_{X'}) \subset \mathcal{O}_{X'}$. Since X' is a hypersurface, $\omega_{X'}$ is invertible. From the fact that X' satisfies the Serre condition S_2, we have

$$\mathcal{C} = \mathrm{Hom}_{\mathcal{O}_{X'}}(\beta_* \mathcal{O}_Y, \mathcal{O}_{X'})$$

(see [R, p. 703]). Assuming Y is Cohen–Macaulay, we have

$$(2.2) \qquad\qquad\qquad \mathcal{C} = \beta_*(\omega_Y) \otimes_{\mathcal{O}_{X'}} \omega_{X'}^{-1}$$

(see [Sz, p. 26]). The following important result is proved in [Z, p. 60]:

Theorem 2.1 (Zariski). *Given that X' is a hypersurface of dimension r, the conductor ideal $\mathcal{C}$ is an unmixed ideal of dimension $r - 1$ in $\mathcal{O}_{X'}$.*

Denote by $C \subset X' \subset \mathbb{P}^5$ the subscheme of pure dimension $r - 1$ defined by $\mathcal{C}$. Let us recall also some basic results from the liaison theory [MP] (see [GLM]). Let $C, D \subset \mathbb{P}^3$ be two locally Cohen–Macaulay curves that are (algebraically) linked

through a complete intersection X of surfaces of degrees s and t (we say $s \times t$ linked) then $\deg C + \deg D = st$ and

$$(2.3) \qquad h^0(\mathcal{I}_C(n)) - h^0(\mathcal{I}_X(n)) = h^2(\mathcal{I}_D(s+t-4-n))$$

$$(2.4) \qquad h^1(\mathcal{I}_C(n)) = h^1(\mathcal{I}_D(s+t-4-n))$$

Moreover given a numerical function f, we let δf denote its first difference function $f(n) - f(n-1)$. Then recall from [S2], [Sl] that $h_C(n) = \delta^2 h^2(\mathcal{I}_C(n))$ is called the spectrum of C. Then $h_C(n) \geq 0$, $\deg C = \sum_{n \in \mathbb{Z}} h_C(n)$, and $p_a(C) = \sum_{n \in \mathbb{Z}} (n-1) h_C(n) + 1$.

If C is obtained from D by an elementary biliaison of height $h = 1$ on a surface of degree s (see [MP, Def. 2.1 III]) then

$$(2.5) \qquad h^0(\mathcal{I}_C(n)) = h^0(\mathcal{I}_D(n-1)) + \binom{n-s+2}{2}$$

$$(2.6) \qquad h_C(n) = h_D(n-1) + h_F(n)$$

where F is a plane curve of degree s.

3. Degree 6

Let us first consider the case $d = 6$. It follows from the exact sequence (2.1) that $h^0(\mathcal{I}_Z) = 1$, thus $\mathcal{I}_Z = \mathcal{O}_{\mathbb{P}^5}$. From [L, Prop. 9.3.43], we infer that $X' \subset \mathbb{P}^5$ is a normal hypersurface (thus X' is Gorenstein) of degree 6 that has rational singularities. Let $(\overline{X}, \overline{H})$ be the Hironaka model of (X, H). Then $|\overline{H}|$ gives a morphism $\rho : \overline{X} \to X'$ that is a resolution of X'. So from [KM, Thm. 5.10], we infer $R^i \rho_*(\mathcal{O}_{\overline{X}}) = 0$ for $i > 0$. In particular,

$$h^2(\mathcal{O}_{\overline{X}}) = h^2(\rho_*(\mathcal{O}_{\overline{X}})) = h^2(\mathcal{O}_{X'}) = 0.$$

However, from Hodge symmetry we infer $h^2(\mathcal{O}_X) = h^0(\Omega_X^2) = 1$. Since the resolution $\overline{X} \to X$ is obtained by a sequence of blow-ups we obtain $h^2(\mathcal{O}_X) = h^2(\mathcal{O}_{\overline{X}})$, a contradiction.

4. Degree 12

From [O, Lem. 4.5] the map φ is then a morphism. Let $X \to Z \to X'$ be the Stein factorization of φ. Since $H \cdot C > 0$ for any curve, the morphism $X \to Z$ is $1 : 1$ so it is an isomorphism. It follows that the normalization of X' is smooth and φ is a finite morphism. Thus $C \subset X'$ is a subscheme supported on the singular locus of X'. From [Ro, Cor. 4.2, Thm. 3.1] the subscheme C is locally Cohen–Macaulay and has pure dimension 3 and degree 36.

Lemma 4.1. *The subscheme $C \subset \mathbb{P}^5$ is arithmetically Buchsbaum.*

Proof. Let us compute $H^i(\mathcal{I}_C(r))$ for $0 < i < 4$ and $r \in \mathbb{Z}$. From (2.2) we deduce that $\varphi_*(\omega_X) = \mathcal{C} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(6)$. Since φ is a finite morphism, the projection formula yields

$$H^i(\mathcal{O}_X(nH)) = H^i((\varphi_*(\mathcal{O}_X))(n)) = H^i((\varphi_*(\omega_X))(n)).$$

So from the Kodaira–Viehweg vanishing theorem we have $H^i(\mathcal{C}(n)) = 0$ for $0 < i < 4$ and $n \neq 7$ and $h^2(\mathcal{C}(6)) = 1$. We conclude using the long cohomology sequence obtained from the following natural exact sequence:

$$(4.1) \qquad 0 \to \mathcal{O}_{\mathbb{P}^5}(-12+n) \to \mathcal{I}_C(n) \to \mathcal{C}(n) \to 0.$$

Analogously we compute that the only non-zero cohomology groups of a hyperplane sections of C are $h^2(\mathcal{I}_{C \cap H}(6)) = h^1(\mathcal{I}_{C \cap H}(7)) = 1$.

Finally, $h^1(\mathcal{I}_{C \cap H_1 \cap H_2}(7)) = 2$ is the only non-zero cohomology group of a codimension 2 linear sections. $\square$

From the exact sequence (4.1), we compute

$$h^0(\mathcal{I}_C(6)) = 1, \ h^0(\mathcal{I}_C(7)) = 6, \ h^0(\mathcal{I}_C(8)) = 21, \text{ and } h^0(\mathcal{I}_C(9)) = 66.$$

From [A, p. 5] we deduce that the resolution of a codimension 2 linear section of C is uniquely determined. Then from the structure Theorem for codimension 2 arithmetically Buchsbaum subscheme from [Ch] we obtain the following minimal resolution (also uniquely determined):

$$0 \to 10\mathcal{O}_{\mathbb{P}^5}(-9) \to \Omega^2_{\mathbb{P}^5}(-6) \oplus \mathcal{O}_{\mathbb{P}^5}(-6) \overset{\alpha}{\to} \mathcal{I}_C \to 0.$$

(we can also find this resolution using Bellinson monades). We infer that, $\mathcal{I}_C$ is generated by one polynomial of degree 6 and ten of degree 9. We can also check that the the cohomology table of an ideal with such a resolution is equal to the cohomology table of $\mathcal{I}_C$. This case will be discussed in a future paper.

Remark 4.2. We can find in this way the possible aCM conductor subschemes of projections of a Calabi-Yau fourfold (and generally) of degree 12. For example the projections of the complete intersections $X_{2,2,3} \subset \mathbb{P}^7$ have conductor loci determined by the maximal minors of a matrix with homogeneous polynomials of degrees

$$\begin{pmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}.$$

It would be interesting to know whether each such subscheme is the conductor locus of a projection of a complete intersection Calabi–Yau threefold of degree 12. We hope that it is possible using this method to find an upperbound for the number of families of Calabi–Yau manifolds of low degrees.

5. Generalities

Consider the diagram

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X' \\ \pi \uparrow & & \uparrow \beta \\ \overline{X} & \overset{\rho}{\longrightarrow} & Y \end{array}$$

where φ is the rational map given by $|H|$, the manifold $\overline{X}$ is the Hironaka model of (X, H) (see [F, (1.4)], [H]), and π is a composition of blow-ups with smooth centers. The composition $\beta \circ \rho$ is the Stein factorization of the birational morphism $\overline{X} \to X$ induced by φ.

Lemma 5.1. *The map φ does not contract surfaces on X to points.*

Proof. Suppose that φ contracts a surface S on X to a point $P \in \mathbb{P}^5$. Let us choose two independent hyperplanes in $\mathbb{P}^5$ passing through P. It follows from [O, Prop. 4.1] that their intersection is an irreducible surface, which is a contradiction since S is its proper component. $\square$

Lemma 5.2. *Assume that the image $\beta(\rho(E))$, where E is the exceptional locus of π, is the sum of a finite number of linear subspaces of $\mathbb{P}^5$ with dimensions ≤ 3. Then the map φ does not contract divisors on X to smaller dimensional subschemes.*

Proof. Suppose that an irreducible divisor D is contracted to a surface $S \subset \mathbb{P}^5$. From Theorem 1.1(2) it follows that there exists a $k \in \mathbb{N}$ such that $D \in |kH|$.

We claim that the surface S is contained in $\beta(\rho(E))$. Indeed suppose there is a curve C that contracts to a point outside $\beta(\rho(E))$. Then C is disjoint with the base locus. Since H is ample we have $C \cdot H > 0$ thus the pre-image of C on $\overline{X}$ cannot be contracted. The contradiction proves the claim.

It follows that S is contained in the sum of linear subspaces $F_1, \ldots, F_s$ of $\mathbb{P}^5$ of dimensions ≤ 3.

If $s = 1$ choose a generic hyperplane $\mathbb{P}^5 \supset R_1 \supset F_1$. Then since $S \subset R_1$ the divisor $H_1 \in |H|$ corresponding to R_1 contains D as a proper component, this is a contradiction with $D \in |kH|$ for some $k \geq 1$.

Suppose that $s > 1$, then since S is irreducible (because D is irreducible), we deduce that S is contained in one of the linear space $F_1, \ldots, F_s$. We obtain a contradiction as before. $\square$

Remark 5.3. The referee observed that the statement of Lemma 5.2 is equivalent to the following: the set

$$\{p \in X' | \dim \varphi^{-1}(p) \geq 1\}$$

has dimension at most 1 (here $\varphi^{-1}(p)$ is the set of $x \in X$ outside the base-scheme such that $\varphi(x) = p$)

Three generic independent elements $H_2, H_3, H_4 \in |H|$ intersect along a sub-scheme S of pure dimension 1. Denote by $[S] \in Z_d(X)$ the fundamental cycle associated to S (as in [Fu, p. 15]). There is a unique decomposition

$$[S] = \Gamma + \Sigma,$$

where Γ and Σ are effective 1-cycles such that

$$\operatorname{supp} \Sigma \subset \operatorname{supp} B$$

where B is the base locus of $|H|$ and $\operatorname{supp} \Gamma$ intersect $\operatorname{supp} \Sigma$ in points. We have

$$12 = \deg(H \cdot (\Gamma + \Sigma))$$

(see [O, §2]). From Theorem 1.1(3), we infer that the Poincaré dual, $cl(\Sigma)$ equals $mh^3/6$. Thus if $H_1 \in |H|$ is generic,

$$(5.1) \qquad\qquad 12 = d + \sum_{p \in \operatorname{supp} B} mult_p(H_1 \cdot \Gamma) + 2m.$$

Remark 5.4. More precisely the above equation holds in the following situation: let $\Theta \subset |H|$ be a 3-dimensional linear subsystem. Then (5.1) holds for arbitrary linearly independent $H_1, \ldots, H_4 \in \Theta$ if Eqnt. (4.0.24) of [O] holds for one set of linearly independent $H_1, \ldots, H_4 \in \Theta$.

Let us prove the following:

Lemma 5.5. *If the base locus of $|H|$ is 0-dimensional, then the generic element of $|H|$ is smooth.*

Proof. (cf. [F, (2.5)]) Let $\pi_1 : T \to X$ be a blow-up of a point from the base locus such that E is the exceptional divisor. Then $\pi^*(H) - sE$ is semi-positive, thus

$$0 \le (\pi^*(H) - sE)^4 = 12 - s^4,$$

so $s = 1$. $\square$

More generally, the referee observed the following:

Lemma 5.6. *If the base locus B of $|H|$ is 0-dimensional, then the intersection D of two generic elements of $|H|$ is smooth.*

Proof. By the Bertini Theorem D is smooth outside B. Let $\Theta \subset |H|$ be a generic 3-dimensional linear subspace and

$$m := 4 - \dim \bigcap_{H' \in \Theta} T_{p_0}(H').$$

We may choose linearly independent $H_1, \ldots, H_4 \in \Theta$ such that

$$T_{p_0}(H_1) \cap \cdots \cap T_{p_0}(H_4) = \bigcap_{H' \in \Theta} T_{p_0}(H').$$

Since $\dim B = 0$ the intersection $H_1 \cap \cdots \cap H_4$ is proper and hence

$$12 = d + \sum_{p \in \text{supp } B} mult_P(H_1 \ldots H_4).$$

Now $d \ge 7$ and hence $mult_P(H_1 \ldots H_4) \le 5$ for all $p \in \text{supp } B$, in particular for $p = p_0$. It follows that $m \ge 2$ and hence the intersection of two generic divisors in Θ is smooth at p_0 (notice that if $d \ge 9$ we actually get $m \ge 3$) $\square$

Remark 5.7. The above argument shows also the following: if $B = B^1 \coprod Z$ where $\dim Z = 0$ there exists a surface containing B which is smooth at each point of Z and hence the scheme B is planar at each of its isolated points. Moreover it is curvilinear (contained in a smooth curve) if $d \ge 9$.

Remark 5.8. Let $\pi \colon \tilde{X} \to X$ the blow-up of the base-scheme B; thus φ defines a regular map $\tilde{\varphi} \colon \tilde{X} \to \mathbb{P}^5$. Let $b \in B$ be an isolated point. If B is a local complete intersection at b then $\pi^{-1}(b)$ is irreducible of dimension 3 and moreover $\tilde{\varphi}(\pi^{-1}(b))$ is a 3-dimensional linear subspace of $\mathbb{P}^5$. In particular if $\dim B = 0$ and B is a local complete intersection we get that the hypothesis of Lemma 5.2 is satisfied and hence for generic D there are no contracted curves on D. Suppose that $d \ge 9$ and $\dim B = 0$; by Remark 5.7 we get that B is curvilinear, in particular a l.c.i..

Let $D \subset X$ be the intersection of two generic divisors from $|H|$ and $X'_D \subset \mathbb{P}^3$ the corresponding linear section of X'. From Theorem 1.1(4) the surface D is reduced and irreducible. By the Bertini theorem (cf. [DH, Thm. 2.1]) the surface D has isolated singularities (from Proposition 5.9 the one dimensional part of the base locus B^1 is reduced at the generic point). We infer that D it is normal (by the Serre criterion). Moreover, D is locally a complete intersection, so it is locally Cohen–Macaulay and $\omega_D = 2H|_D$. If $H_1, H_2, H_3 \in |H|$ are generic we write as above $[H_1 \cap H_2 \cap H_3] = \Gamma' + \Sigma'$ where Σ' and Γ' are Weil divisors such

that $\operatorname{supp}\Sigma' \subset \operatorname{supp}B$ and $dim(\operatorname{supp}\Gamma' \cap \operatorname{supp}\Sigma') = 0$. The following diagram is induced from diagram (5):

$$
\begin{array}{ccc}
D & \xdashrightarrow{\;|H'_D|\;} & X'_D \subset \mathbb{P}^3 \\
{\scriptstyle \pi_D}\Big\uparrow & & \Big\uparrow {\scriptstyle \beta_D} \\
\overline{D} & \xrightarrow{\;\;\rho_D\;\;} & Y_D
\end{array}
$$

where H'_D is the restriction of H to D. Denote by H_D the pull back of $\mathcal{O}_{X'_D}(1)$ by $\beta_D \circ \rho_D$.

Finally consider the following general proposition:

Proposition 5.9. *Suppose that $d \geq 7$. The base-scheme B is reduced at the generic point of any of its 1-dimensional irreducible components.*

Proof. If the proposition is not true the cycle Σ associated to generic $H_2, H_3, H_4 \in |H|$ is non-reduced. By equation (5.1) one gets that $d = 7$, $\Sigma = 2\Sigma'$ where Σ' is an irreducible curve, B is of pure dimension 1, non-reduced irreducible, moreover there is a unique point $p \in \Sigma \cap \Gamma$ and

$$(5.2) \qquad\qquad mult_p(H_1 \cdot \Gamma) = 1 \quad H_1 \in |H| \quad \text{generic.}$$

We claim that for generic $H_2, H_3, H_4 \in |H|$ we have

$$(5.3) \qquad\qquad (T_pH_2) \cap (T_pH_3) \cap (T_pH_4) = T_pB \qquad \forall p \in B.$$

First notice that

$$(5.4) \qquad\qquad \dim T_pB \geq 2 \qquad \forall p \in B$$

because B is everywhere non-reduced and of dimension 1; moreover $\dim T_pB = 2$ for generic $p \in B$ because $\Sigma = 2\Sigma'$. Let $H_3, H_4 \in |H|$ be generic; then

(a) $dim((T_pH_3) \cap (T_pH_4)) = 2$, if $\dim((T_pH_4) = 3$ (thus for generic p),
(b) $dim((T_pH_3) \cap (T_pH_4)) = 4$, only if $\dim((T_pB) = 4$.

Now choose a generic $H_2 \in |H|$; then (5.3) follows from (a) and (b) above and (5.4). Let $[H_2 \cap H_3 \cap H_4] = \Sigma + \Gamma$, as usual, and p be the unique point in $\Sigma \cap \Gamma$. Let $H_4 \in |H|$ be arbitrary; then $T_pH_1 \supset T_pB$ by (5.3). It follows that $mult_p(H_1 \cdot \Gamma) \geq 2$. We obtained a contradiction with (5.2). $\qquad\square$

Corolary 5.10. *Suppose that $d \geq 7$. Let $H_2, H_3, H_4 \in |H|$ be generic and $[H_2 \cap H_3 \cap H_4] = \Sigma + \Gamma$ where Σ is supported in B and $\operatorname{supp}(\Gamma) \cap supp(\Sigma)$ is 0-dimensional. Then the cycle Σ is reduced.*

Proof. The surface $D = H_2 \cap H_3$ is reduced, irreducible and normal. Then $\overline{\Gamma} + \overline{\Sigma}$ is the Cartier divisor on D corresponding to H_4 (such that $\operatorname{supp}\Sigma = \operatorname{supp}\overline{\Sigma}$ and $\operatorname{supp}\Gamma = \operatorname{supp}\overline{\Gamma}$). It is enough to prove that the scheme $\overline{\Sigma}$ is reduced at his generic point q. Suppose the contrary, then $T_q\overline{\Sigma} = T_qD$ and this holds for a generic H_4. This is a contradiction since B is reduced at q. $\qquad\square$

Remark 5.11. Suppose that Γ' and Σ' are Cartier divisors (for $H_2, H_3, H_4 \in |H|$ generic). Then the linear system $|\Gamma'| + \Sigma'$ can be naturally identified with the linear system $|H||_D$ being the restriction of $|H|$ to D (here $D = H_1 \cap H_2$ where $H_1, H_2 \in |H|$ are generic, recall that D is normal). Indeed, it is enough to observe that they have the same dimension. This follows from the fact that $H^1(\mathcal{O}_{H_1}(H)) = 0$, since

H is ample. Thus $|H||_D$ is a complete linear system. It follows from Corollary 5.10 that the 1-dimensional part of the base locus of $|H||_D$ is reduced and equal to Σ.

6. Degree 11

It follows from (5.1) that if $d = 11$ then $m = 0$. We infer that φ has 0-dimensional base locus. Moreover, $\operatorname{supp} B$ is exactly one point P such that $\operatorname{mult}_P(H_1 \cdot \Gamma) = 1$.

Lemma 6.1. *With the above assumptions the morphism $\pi : \overline{X} \to X$ is the blowing-up of P.*

Proof. Let E' be the exceptional divisor of π', the blowing-up of X at P. By Lemma 5.5 it is enough to show that $L := (\pi')^*(H) - E'$ is base-point-free on E'. Since $L|_{E'} = \mathcal{O}_{\mathbb{P}^3}(1)$, the base locus is a linear space. Moreover, from $\operatorname{mult}_P(H_1 \cdot \Gamma) = 1$ the tangent space to Γ corresponds to a point outside this base locus. Since the curve Γ is smooth at P and is the intersection of three general elements of $|H|$, we conclude that the base locus is empty. $\square$

We conclude also that $\beta_D(\rho(E'))$ is a linear space of dimension ≤ 3. Thus from Lemma 5.2 we can choose the smooth surface D in such a way that the morphism ρ_D (from diagram 5) does not contract curves (it is enough to choose the divisors defining D such that the corresponding hyperplanes in $\mathbb{P}^5$ meets along a linear space disjoint from the image of the curves contracted by φ), so it is an isomorphism and $\overline{D} = Y_D$. Let $\mathcal{C}$ be the conductor of the normalization $Y_D \to X'_D$, as before we deduce that C is locally Cohen–Macaulay and of pure dimension 1. The degree of $C \subset \mathbb{P}^3$ is

$$\frac{1}{2}(77 - K_{Y_D} \cdot H_D) = 26.$$

Let us compute

$$H^i(((\beta_D \circ \rho_D)_*(\omega_{\overline{D}}))(n))$$

(as in the proof of Lemma 4.1). From the projection formula this cohomology group has dimension equal to

$$h^i(\overline{D}, K_{\overline{D}} + (\beta_D \circ \rho_D)^*(\mathcal{O}_{X'_D}(n))).$$

From the Kawamata–Viehweg theorem the last number is 0 for $i = 1$ and $n \geq 0$.

We see that $K_{\overline{D}} = 3E + 2H_D$, where E is the reduced exceptional locus of π_D and

$$H_D = \pi_D^*(H'_D) - E = (\beta_D \circ \rho_D)^*(\mathcal{O}_{X'_D}(1)).$$

Next, as in the proof of Lemma 4.1,

$$h^i(\mathcal{C}(n)) = h^i(\overline{D}, K_{\overline{D}} + (n - 7)H_D)) = h^i(3E + (n - 5)H_D)).$$

From the exact sequence

$$0 \to \mathcal{O}_{\overline{D}}(2E + (n - 5)H_D) \to \mathcal{O}_{\overline{D}}(3E + (n - 5)H_D) \to \mathcal{O}_E(3E + (n - 5)H_D) \to 0$$

and the fact that $3E + (n - 5)\tilde{H}|_E = \mathcal{O}_E(n - 8)$ we obtain, using [L, Lem. 4.3.16],

$$h^0((n - 5)(H_D + E)) = h^0(X, \mathcal{O}((n - 5)H)) = h^0(3E + (n - 5)H_D)$$

for $n = 6, 7, 8$. Hence $h^0(\mathcal{I}_C(5)) = 1$, $h^0(\mathcal{I}_C(6)) = 4$, $h^0(\mathcal{I}_C(7)) = 11$. Moreover

$$h^0(\mathcal{I}_C(7 + n)) = h^0(K_{\overline{D}} + nH_D) + h^0(\mathcal{O}_{\mathbb{P}^3}(n - 4))$$

for $n > 0$.

Let us compute $h^2(\mathcal{I}_C(n))$. We have $h^2(\mathcal{C}(n)) = h^2(\overline{D}, K_{\overline{D}} + (n-7)H_D)) = h^0((7-n)H_D)$. Thus $h^2(\mathcal{C}(n)) = 0$ for $n > 7$ and $h^2(\mathcal{C}(k+7)) \leq h^0(D, kH)$ for $k \geq 0$. Consider the following long cohomology exact sequence obtained like in (4.1):

(6.1) $0 \to h^2(\mathcal{I}_C(n)) \to h^2(\mathcal{C}(n)) \to h^3(\mathcal{O}_{\mathbb{P}^3}(-11+n)) \to h^3(\mathcal{I}_C(n)).$

Since $h^3(\mathcal{I}_C(n)) = h^3(\mathcal{O}_{\mathbb{P}^3}(n))$ from Serre duality we obtain

$$h^2(\mathcal{I}_C(n)) = h^2(\mathcal{C}(n)) - h^0(\mathcal{O}_{\mathbb{P}^3}(7-n)) + h^0(\mathcal{O}_{\mathbb{P}^3}(-4-n)).$$

By the long exact sequence (6.1) we infer $h^2(K_{\overline{D}}) = 1$, $h^2(K_{\overline{D}} + H_D) = 4$. Consider the following table where the last column is computed using the Riemann-Roch theorem $\chi(K_{\overline{D}} + nH_D) = \frac{11}{2}n^2 + \frac{25}{2}n + 12$.

TABLE 1.

n	$h^0(K_{\overline{D}} + nH_D)$	$h^1(K_{\overline{D}} + nH_D)$	$h^2(K_{\overline{D}} + nH_D)$	$\chi(K_{\overline{D}} + nH_D)$
0	11	0	1	12
-1	4	3	4	5
-2	1	y	$8+y$	9
-3	0	z	$24+z$	24

We have $3 \geq y \geq 2$. From the table below we obtain also the cohomology table $h^i(\mathcal{I}_C(n))$.

In particular we infer $h^2(\mathcal{I}_C(6)) = h^1(\mathcal{I}_C(7)) = 0$ thus I_C is generated in degree ≤ 8 (see [BM, Lemm. 1.2]). So I_C is generated by one generator of degree 5 one of degree 7 and at least six generators of degrees 8. Let $B \subset \mathbb{P}^3$ be a Cohen–Macaulay curve 5×8 linked to C.

n	$h^0(\mathcal{I}_B(n))$	$h^1(\mathcal{I}_B(n))$	$h_B(n)$
6	$19 + t$	t	0
5	$z + 5$	z	0
4	$y - 2$	y	1
3	0	3	7
2	0	0	3
1	0	0	2

It follows from [Sl, Thm.1.1(1)] that the Rao module $M_B := \oplus_{n \in \mathbb{Z}} H^1(\mathcal{I}_B(n))$ is generated in degree 3.

The Betti table $\beta(M_B) = (\beta_{i,j})$ of the minimal resolution of M_B (see [E]) are as follows.

$j \setminus i$	0	1	2	$\cdots$
3	3	$12 - y$	$18 - 4y + z + k$	$\cdots$
4	0	k	d	$\cdots$
$\cdots$	0	$\cdots$	$\cdots$	$\cdots$

Lemma 6.2. *The Betti numbers $\beta_{1,j}$ from the table above are 0 for $j \geq 6$.*

Proof. First consider a minimal free resolution of M_C the Rao module of C (it is appropriately dual to the resolution of M_B see [MP, p. 39])

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \to L_0 \to M_C \to 0.$$

We have to prove that L_3 have no summand of degree ≥ -7.

It is known that I_C has a minimal resolution of the form

$$0 \to L_4 \to L_3 \oplus F \to \mathcal{O}_{\mathbb{P}^3}(-5) \oplus \mathcal{O}_{\mathbb{P}^3}(-7) \oplus (6+s)\mathcal{O}_{\mathbb{P}^3}(-8) \to I_C \to 0,$$

where $s \geq 0$ and F is locally free. On the other hand we compute the function $r_C(n) = \delta(\delta^3 h^0(\mathcal{I}_C(n)) - \binom{n}{0})$ and obtain $r_C(5) = r_C(7) = 1$, $r_C(8) = 6$, $r_C(9) = -10$, and $r_C(10) = 3$. We now deduce from [MP, p. 50] that if L_3 has a component of degree ≥ -7 then L_4 has also such a component. This is a contradiction since $L_4 = 3\mathcal{O}_{\mathbb{P}^3}(-10)$. $\qquad\square$

Let M be the module M_B shifted to 0 (such that it is generated in degree 0). Let us find the invariant h from [MP] of the minimal curve for the biliaison class associated to the module M (see [GLM, p. 287]). We have $3 \geq h$ since M_B is generated in degree 3. We shall use several times the following:

$$(6.2) \qquad h + \deg L_1 - \deg L_0 = h + 12 - y + 2k = \sum_{n \in \mathbb{Z}} n \cdot q(n).$$

Here $q \colon \mathbb{Z} \to \mathbb{Z}$ is a function defined in [MP] (see [GLM, p. 287]) related to the minimal curve in the biliaison class. If $y = 2$ then $b_{2,5} = 10 + z + k$. If $h = 2$ then B can be bilinked down to B_0. From (2.6) we infer $s = 5$. Thus $q(2) = 6 + k$ so form (6.2) $q(n) = 0$ for $n \neq 2$. In case $h < 2$ we obtain a contradiction with (6.2) since $q(2) \geq 6 + k$. When $h = 3$ then $5 + z = 10 + z + k - q(2)$ thus $q(3) = 1$

If $y = 3$ then $b_{2,5} = 6 + z + k$. We have $h < 3$ since $h^0(\mathcal{I}_B(4)) = 1 = a_0 + h$. Assume $h = 2$ then B is bilinked down to B_0 on the quartic (from (2.5)) thus $q(2) = 4 + k$ and $q(3) = 1$ the last case in when $h = 1$ and $q(2) = 5 + k$.

Problem 6.3. *Does there exist a module M_B with the given invariants?*

Remark 6.4. If a module M_B with a resolution that begin as above with the given q function exists such that the invariant $a_1 \leq 5$ [MP, Def. 2.4IV], then from [MP] a curve with cohomology Table 1 exists. Observe that $q(2)$ is an invariant equal to $\inf(\alpha - 1, \beta)$ where α is the rank of the second map σ_2 in the linear strand (see [E]) of the resolution of M_B and $\beta = \max\{r \in \mathbb{N}$ the r-minors of σ_2 are not all 0 and are coprime$\}$ (thus $q(2)$ depend only of the first two grading of the Rao module and the map between them). We suspect that such modules do not exist.

7. Degree 10

In the case $d = 10$ the situation becomes more complicated.

Lemma 7.1. *The base locus of $|H|$ is 0-dimensional.*

Proof. Suppose that $m \neq 0$ in (5.1). It follows that $m = 1$ and

$$\sum_{p \in \operatorname{supp} B} \operatorname{mult}_p(H_1 \cdot \Gamma) = 0.$$

But this is impossible since the intersection of ample divisors defining $\Gamma + \Sigma$ is connected, thus $\operatorname{supp} \Gamma \cap \operatorname{supp} \Sigma \neq \varnothing$. $\qquad\square$

It follows from (5.1) that B has length 2.

Lemma 7.2. *Suppose that* $\operatorname{supp} B$ *is one point. Then the system* $\gamma^*(H) - E_1|_{E_1}$ *is the system of hyperplanes passing through one point* Q. *The pair* (X_2, H_2), *where* H_2 *is the proper transform of* $|H|$, *obtained by the blowing-up of* Q, *is a Hironaka model of* (X, H) *(i.e. the linear system* $|H_2|$ *is base-point-free).*

Proof. First from Lemma 5.5 the restriction of the system $\gamma^*(H) - E_1|_{E_1}$ is a linear subsystem of $|\mathcal{O}_{E_1}(1)|$. Thus it is a linear system of hyperplanes passing through a linear subspace $\Lambda \subset E_1$.

We claim that the dimension of Λ is 0. Indeed, suppose it is larger. Then Γ has to be singular. It follows that $\operatorname{mult}_p(H_1 \cdot \Gamma) \geq 4$ since Γ is tangent to H_1.

Now, since $\operatorname{mult}_p(H_1 \cdot \Gamma) = 2$ the blowing-up of Q separates the proper transforms of Γ and H_1. $\qquad\square$

Lemma 7.3. *The morphism induced by* φ *on* D *does not contract curves.*

Proof. It is enough to prove that $\beta(\rho(E))$ is a sum of linear spaces (i.e. the assumptions of Lemma 5.2 holds). If B has two reduced components then we argue twice as in the case of degree 11. If $\operatorname{supp} B$ is one point then we use Lemma 5.2. $\qquad\square$

Suppose that $\operatorname{supp} B$ is one point, we obtain the following:

Lemma 7.4. *The morphism* ρ_D *is* $1:1$ *on* $\overline{D} - E$, *where* E *is the strict transform of the exceptional curve of the first blow-up on* $\overline{D}$. *Moreover,* ρ_D *contracts* E *to a Du Val singularity of type* A_1 *on* X'_D.

Proof. The system $|H'_D|$ (the restriction of $|H|$ to D) does not contract curves. Now, E is a smooth rational curve with self intersection -2, that is contracted by ρ_D to a normal singularity. This singularity must be an ordinary double point. $\qquad\square$

We deduce that Y_D is locally Cohen–Macaulay (and ω_{Y_D} is locally free). From Theorem 2.1 the ideal defined by the conductor $C \subset X'_D$ has pure dimension 1. Since K_{Y_D} is a Cartier divisor, and $C \in |6L - K_{Y_D}|$ we deduce that C is locally Cohen–Macaulay (from the proof of [Ro, Thm. 3.1]). Denote $L := \beta^*(\mathcal{O}_{X'_D}(1))$. Then from [R, Prop. 2.3] we infer

$$\beta_D^*(\mathcal{O}_{X'_D}(6)) = \beta_D^*(K_{X'_D}) = K_{Y_D} + C_1,$$

where $C_1 \subset Y_D$ is the Cartier divisor defined by the conductor.

We compute using [KLU, Thm. 3.5] (since Y_D is Cohen–Macaulay, the normalization is locally flat of codimension 1) that $2\deg(C) = \deg(C_1)$.

Now, Y_D has rational singularities it is $\mathbb{Q}$-factorial thus we can compute as follows $\deg(C_1) = L(6L - K_{Y_D})$. So

$$2\deg(C) = 6H_D^2 - H_D\rho_D^*(K_{Y_D}) = 34,$$

since $H_D \cdot K_{\overline{D}} = 26$.

Let us compute $h^i(\mathcal{I}_C(n))$ for $0 \leq i \leq 2$. From the proof of Lemma 4.1 it is enough to find

$$H^i((\beta_D)_*(\omega_{Y_D})(n)) = H^i(K_{Y_D} + nL).$$

From [EV, Cor. 6.11] we have

$$R^j(\rho_D)_*(K_{\overline{D}} + nH_D) = 0$$

for $j > 0$. Thus using the Leray spectral sequence, we infer

$$H^i(K_{\overline{D}} + nH_D) = H^i((\rho_D)_*(K_{\overline{D}} + nH_D))$$

for $i \geq 0$. By the projection formula and the fact that Y_D has rational singularities we infer

$$h^i(K_{\overline{D}} + nH_D) = h^i(K_{Y_D} + nL).$$

From Kawamata–Viehweg theorem the last number is 0, for $i = 1$, and $n \geq 0$. Let us find

$$h^0(\mathcal{I}_C(n)) - h^0(\mathcal{O}_{\mathbb{P}^3}(n - 10)) = h^0(K_{Y_D} + (n - 6)H_D).$$

Denote by F_D the second exceptional divisor of π_D, then $\pi_D^*(H_D') = H_D + E_D + 2F_D$ and $K_{\overline{D}} = 2H_D + 6F_D + 3E_D$. Let us compute $h^0(6F_D + 3E_D + (n - 4)H_D)$.

We claim that

$$h^0(6F_D + 3E_D + (n-4)H_D) = h^0((n-4)(2F_D + E_D + H_D)) = h^0(D, (n-4)H_D')$$

for $n = 4, 5, 6, 7$. The last equality follows from [L, Lem. 4.3.16], for the first we argue as in Section 6. After the first blow-up $\gamma_D : D_1 \to D$ with exceptional divisor E_1 and H_1 the strict transform of H, we obtain

$$h^0(3E_1 + (n - 4)H_1) = h^0((n - 4)(E_1 + H_1)).$$

To conclude we use again [L, Lem. 4.3.16], and the long exact sequence as in Section 6. Finally, $h^0(D, (n-4)H_D') = h^0(X, (n-4)H) + h^0(X, (n-6)H) - 2h^0(X, (n-5)H)$, so we obtain $h^0(\mathcal{I}_C(4)) = 1$, $h^0(\mathcal{I}_C(5)) = 4$, $h^0(\mathcal{I}_C(6)) = 11$, $(h^0(\mathcal{I}_C(7)) = 30)$. Now, if $\mathrm{supp}\, B$ is two points then ρ_D is a isomorphism and we argue as in the case $d = 11$ (and obtain the same result as in this case). We obtain the following table with $\chi(K_{Y_D} + nL) = \frac{10}{2}n^2 + \frac{26}{2}n + 12$:

n	$h^0(K_{Y_D} + nL)$	$h^1(K_{Y_D} + nL)$	$h^2(K_{Y_D} + nL)$	$\chi(K_{Y_D} + nL)$
0	11	0	1	12
−1	4	4	4	4
−2	1	a	$5 + a$	6
−3	0	$2 + x$	$20 + x$	18

where as before $6 \geq a \geq 5$ and $x \geq 2$. Let $B \subset \mathbb{P}^3$ be a degree 11 curve 4×7 linked to C. We obtain the following cohomology table:

n	$h^0(\mathcal{I}_B(n))$	$h^1(\mathcal{I}_B(n))$	$h_B(n)$
5	$y + 9$	y	0
4	$x + 1$	$x + 2$	0
3	$a - 5$	a	1
2	0	4	7

Suppose first that $a = 6$, then the Betti table of the minimal resolution of M_B is as follows.

$j \setminus i$	0	1	2	$\ldots$
2	4	10	$2+x+k$	$\ldots$
3	0	k	d	$\ldots$
$\ldots$	0	0	$\ldots$	$\ldots$

Since $h^0(\mathcal{I}_B(3)) > 0$ we have $h < 2$. If $h < 1$ then $q(2) \geq 4 + k$ so $h = 0$ and $q(2) = 5 + k$. Finally if $h = 1$ then from (2.5) we infer $\deg B_0 = 8$, $q(2) = 4 + k$ thus $q(3) = 1$.

Assume that $a = 5$, then $b_{1,1} = 11$ and $b_{2,2} = 6 + k + x$. It follows that either $h = 2$, $q(2) = 5 + k$, and $q(3) = 1$ or $h = 1$ and $q(2) = 6 + k$.

8. Degree 9

If $H_1, H_2, H_3 \in |H|$ are generic we write $[H_1 \cap H_2 \cap H_3] = \Gamma' + \Sigma'$ where $\operatorname{supp} \Sigma' \subset \operatorname{supp} B$ and $dim(\operatorname{supp} \Gamma' \cap \operatorname{supp} \Sigma') = 0$. Denote as above by D the intersection of H_1 and H_2 and by $X'_D \subset \mathbb{P}^3$ the corresponding linear section of X'. Recall that the surface D is reduced, irreducible, and normal. Moreover, D locally Cohen–Macaulay and $\omega_D = 2H|_D$.

• Suppose that the base locus has dimension 1, i.e. $\Sigma' \neq 0$. Then by [O, Prop. 5.4] we deduce that the cycle Σ' is a reduced irreducible local complete intersection curve and is the scheme-theoretic base locus of $|H||_D$. By [DH, Thm. 2.1] the surface D has only isolated singularities at singular points of Σ'. From the proof of [O, Prop. 5.4] the curves Γ' and Σ' intersect transversally in one point that varies on B^1 (when H_3 changes see [O, (5.3.16)]). It follows that Γ' and Σ' are Cartier divisors on D. Then we infer using (5.1) that the linear system $|\Gamma'|$ has no base points (the morphism given by the linear system $|\Gamma'|$ is equal to $\varphi|_D$ see Remark 5.11).

Lemma 8.1. *The morphism induced by φ on D does not contract curves.*

Proof. Suppose that the curve C is contracted. Assume moreover, that C intersects Σ' in a smooth point p of Σ' (then we have $mult_p((\Sigma' + \Gamma') \cdot C) = mult_p(H_1 \cdot C)$). From $cl(C) = mh^3/6$ we have $2 \leq H_1 \cdot C = (\Gamma' + \Sigma') \cdot C$. Since, Γ' and Σ' are Cartier divisors we infer $C \cdot \Sigma' \geq 2$. But $\Gamma' - C$ is effective, this is a contradiction with $\Sigma' \cdot \Gamma' = 1$.

Let us consider the remaining case where there is a point $P_0 \in \Sigma'$ such that for a generic choice of D we have that $P_0 \in C$, where C is a contracted curve. Let $\pi : \tilde{X} \to X$ be the blow up of $\Sigma' \subset X$ (note that $\tilde{X}$ can singularities at points in the pre-image by π of singular points of Σ'). Then each fiber of the exceptional divisor $E \to \Sigma'$ map by $|\pi^*(H) - E|$ to a plane in $\mathbb{P}^5$. It follows that the image of curves contracted by φ is contained in a 2-dimensional linear subspace of $\mathbb{P}^5$ (the image of the fiber of E that maps to P_0). We conclude as in the proof of Lemma 5.2. $\square$

It follows that $|\Gamma'|$ gives the normalization of the linear section $X'_D \subset \mathbb{P}^3$ of X'.

From [R, Prop. 2.3], we have $5\Gamma' = 2\Gamma' + 2\Sigma' + C_1$ where $C_1 \subset D$ is the Cartier divisor defined by the conductor. Finally, using [KLU, Thm. 3.5], we compute that the degree of the conductor subscheme $C \subset X'_D \subset \mathbb{P}^3$ is $\frac{1}{2}(\Gamma'(3\Gamma' - 2\Sigma')) = \frac{25}{2}$ and obtain a contradiction.

• Assume now that the base locus is 0-dimensional, i.e. $\Sigma' = 0$. Let us define the surface D as above. From Lemma 5.6 the surface D is smooth.

We claim that Γ' does not contract curves. Indeed, if $\text{supp } B$ are two or three points, we argue as in the cases of degrees 10 and 11. If $\text{supp } B$ is one point P then each contracted curve contain P, we conclude as in the proof of Lemma 5.2. The claim follows.

Let D' be the surface obtained from D by blowing-up P. Denote by Γ'' the strict transform of Γ' on D'. We show as in the proof of Lemma 5.6 that the generic element of $|\Gamma'|$ is smooth at P. Thus $(\Gamma'')^2 = 11$ and Γ'' has exactly one base point P' on the exceptional divisor E' on D', moreover $\Gamma'' \cdot E' = 1$. Blowing-up P' we obtain a surface D'' with exceptional divisor E'' such that Γ''' (resp. E'') is the strict transform of Γ'' (resp. E'). We have $(\Gamma''')^2 = 10$, the linear system $|\Gamma'''|$ has exactly one base point $P'' \in E''$, and $\Gamma''' \cdot E'' = 1$. The strict transform H_D of Γ' on $\overline{D}$, the blowing-up of D'' at P'' gives a base-point-free linear system; denote by $\pi_D : \overline{D} \to D$ the composed morphism. The morphism ρ defined by $|nH_D|$ for n large enough has normal image Y_D and contracts only the strict transforms of E' and E''. It follows from the Stein factorization theorem that Y_D is the normalization of the chosen codimension 2 linear section X'_D of $X' \subset \mathbb{P}^5$.

We have two possibilities: either $P'' \in E'' - E'$, or $P'' \in E'$. In the first case we infer from the Artin contraction theorem that Y_D has exactly one singular point being a Du Val singularity of type A_2. In particular, Y_D is locally Cohen–Macaulay, and ω_{Y_D} is locally free. Thus the conductor defines a subscheme $C \subset X'_D$ that is of pure dimension 1 and locally Cohen–Macaulay.

Next, if $P'' \in E'$ then E' and E'' are disjoint -3 and -2 curves on $\overline{D}$. We find that Y_D has exactly two singular points: a quotient singularity of type $\frac{1}{3}(1,1)$ (see [KM, Rem. 4.9]) and a Du Val singularity of type A_2. It follows from [KM, Prop. 5.15] that Y_D has rational singularities (thus is locally Cohen–Macaulay) and is $\mathbb{Q}$-factorial. We infer that the conductor of the normalization $\beta_D : Y_D \to X'_D$ defines a subscheme $C \subset X'_D$ that is of pure dimension 1 and locally Cohen–Macaulay.

Remark 8.2. The referee observed that the possibility $P'' \in E'$ can be considered using the fact that the base scheme is a curvilinear scheme of length 3 (see Remark 5.8).

As before we deduce from (2.2) that

$$h^1((\beta_D)_*(\omega_{Y_D})(n)) = h^1(\mathcal{I}_C(n+5)).$$

Define $L := \beta_D^*(\mathcal{O}_{X'_D}(1))$. Then since Y_D has rational singularities, we infer from [EV, Cor. 6.11] that

$$h^i(K_{Y_D} + nL) = h^i(K_{\overline{D}} + nH_D).$$

Now, let us compute $h^0(K_{\overline{D}} + nH_D) - h^0(\mathcal{O}_{\mathbb{P}^3}(-4+n)) = h^0(\mathcal{I}_C(n+5))$. Denote by F the exceptional divisor of the last blow-up. Then $K_{\overline{D}} = \pi_D^*(K_D) + E' + 2E'' + 3F$ and $\pi_D^*(\Gamma') = H_D + E' + 2E'' + 3F$.

We claim that

$$h^0((n+2)H_D + 3E' + 6E'' + 9F) = h^0(K_{\overline{D}} + nH_D) = h^0(K_D + n\Gamma')$$

for $n = -2, -1, 0, 1$. From [L, Lem. 4.3.16] we have

$$h^0((n+2)H_D) = h^0((n+2)\Gamma') = h^0((n+2)(H_D + E' + 2E'' + 3F)).$$

The claim follows using the long exact sequence as in Section 6. We infer $h^0(\mathcal{I}_C(3)) = 1$, $h^0(\mathcal{I}_C(4)) = 4$, $h^0(\mathcal{I}_C(5)) = 11$, (and $h^0(\mathcal{I}_C(6)) = 30$).

Finally, using [KLU, Thm. 3.5], we compute the degree

$$2deg(C) = L(5L - K_{Y_D}) = 45 - \rho^*(K_{Y_D}) \cdot H_D = 45 - H_D \cdot (K_{\overline{D}} + aE' + bE'') =$$

$$= 45 - H_D \cdot K_{\overline{D}} = 45 - (\pi_D^*(2\Gamma') + E' + 2E'' + 3F)(\pi_D^*(\Gamma') - E' - 2E'' - 3F) = 18.$$

In this case $\chi(K_{Y_D} + nL) = \frac{9}{2}n^2 + \frac{27}{2}n + 12$ and

n	$h^0(K_{Y_D} + nL)$	$h^1(K_{Y_D} + nL)$	$h^2(K_{Y_D} + nL)$	$\chi(K_{Y_D} + nL)$
0	11	0	1	12
-1	4	5	4	3
-2	1	y	$2 + y$	3
-3	0	z	$12 + z$	12
-4	0	t	$30 + t$	30

as before $9 \geq y \geq 8$, $z \geq 8$, and $t \geq 5$. Let $B \subset \mathbb{P}^3$ be a degree 9 with $p_a(B) = 1$ curve 3×6 linked to C.

n	$h^0(\mathcal{I}_B(n))$	$h^1(\mathcal{I}_B(n))$	$h_B(n)$
4	$t - 1$	t	0
3	$z - 7$	z	0
2	$y - 8$	y	1
1	0	5	7

First if $y = 9$ then $h_C(5) = 1$, it follows from [Sl, Thm. 1.1] that $h_C(4) = h_C(3) = 1$ (see the 1-property [S3]), thus $z = 11$ and $t = 11$. We can use [S2, Cor.4.4] to show that B is not minimal. But B can be bilinked down (with height -1) on the quadric to the minimal curve B_0 (of degree 7). We compute form Equation (2.5) that $h^0(\mathcal{I}_{B_0}(2)) = 1$ and $h^0(\mathcal{I}_{B_0}(3)) = 4$. The Betti table of the minimal resolution of M_B is as follows.

$j \backslash i$	0	1	2	$\dots$
1	5	11	$5 + k$	$\dots$
2	0	k	d	$\dots$
$\dots$	0	0	$\dots$	$\dots$

We infer that $q(2) = 4 + k$ thus $q(3) = 1$ since $h = 0$.

Assume that $y = 8$, the Betti table of the minimal resolution of M_B is as follows.

$j \backslash i$	0	1	2	$\dots$
1	5	12	$z - 2 + k$	$\dots$
2	0	k	d	$\dots$
$\dots$	0	0	$\dots$	$\dots$

If B is minimal in its biliaison class then $q(2) = 5 + k$ thus $q(3) = 1$ since $h = 1$. We can find a bound of the invariant a_1 from [MP, p. 77]. This gives an evidence for the conjecture since a_1 is different than expected.

Lemma 8.3. *The invariant $a_1 > 3$.*

Proof. Suppose the contrary i.e. $a_1 = 3$ then from [MP, Prop. 5.10IV] B is 3×4 linked to a minimal curve C_0 from the class of C. Then $\deg C_0 = 3$ and $p_a(C_0) = -8$, $h^0(\mathcal{I}_{C_0}(2)) = 0$, $h^0(\mathcal{I}_{C_0}(3)) = 2$. Since C_0 is not extremal we find $e(C_0) = -2$ thus C_0 has a quasi-primitive structure supported on a sum of lines [Sl, Rem. 3.5] and non reduced [Sl, Ex. 2.11]. If C_0 is supported on two lines then we obtain a contradiction with $h^0(\mathcal{I}_{C_0}(3)) = 2$ from the proof of [N1, Prop. 3.3] (and by [N1, Prop. 3.2] since C_0 is not extremal). If it is supported on one line then the possible number of cubic generators of $\mathcal{I}_{C_0}$ are computed in [N1, Rem. 2.4, Prop. 2.1], a contradiction. $\square$

If B is not minimal it can be bilinked down (on a cubic) to a minimal curve B_0. From (2.5) we deduce that $q(2) \geq 6 + k$ thus $h = 0$, $q(2) = 6 + k$, and $a_1 > 3$.

9. Degrees 8 and 7

9.1. Suppose first that the base locus B is 1-dimensional. If $H_1, H_2, H_3 \in |H|$ are generic we write as usual $H_1 \cap H_2 \cap H_3 = \Gamma' + \Sigma'$. Then the cycle Σ' is not zero (and reduced). Denote as above by D the intersection of H_1 and H_2.

Claim 9.1. *Let $\Theta \subset |H|$ be a generic 3-dimensional linear subsystem (that satisfy Remark 5.4). Given a generic $p \in \operatorname{supp} B^1$ (where B^1 is the union of 1-dimensional components of the base scheme B) there exist linearly independent $H_2', H_3', H_4' \in \Theta$ such that $p \in \operatorname{supp} \Gamma$, where $\Sigma + \Gamma = [H_2' \cap H_3' \cap H_4']$ are as usual. Suppose that there exists $p_0 \in \operatorname{supp} B^1$ such that one has $p \in \operatorname{supp} \Gamma$ for a generic set of linearly independent $H_2', H_3', H_4' \in \Theta$, where $\Sigma + \Gamma = [H_2' \cap H_3' \cap H_4']$ is as usual. Then there is a unique such p_0, moreover $d = 7$ and for a generic set of linearly independent $H_2', H_3', H_4' \in \Theta$ the corresponding cycle Σ is reduced and irreducible.*

Proof. Let us consider the first statement. We can assume that $p \in B^1$ is smooth. Let us show that there are three independent elements of Θ such that their intersection is singular at p. Since Θ is 3-dimensional, we can find two elements H_2', H_3' of it that have the same tangent space at p. Their intersection D_1 is singular at p (and from Theorem 1.1(4) irreducible and reduced). It is enough to choose the third element H_4' generically (such that H_4' cuts D_1 transversally at a generic point of B^1).

Let us prove the second statement. Let $\Theta = \mathbb{P}(W)$ where $W \subset H^0(X, \mathcal{O}_X(H))$ is a 4-dimensional sub vector-space. Given $p \in B^1$ consider the differential map

$$\delta_p \colon W \to \Omega_p X, \qquad \delta_p(\sigma) = d\sigma(p).$$

Let $K_p := \ker(\delta_p)$. Let p be a generic point of a component of B^1; then $dim K_p = 1$ by Proposition 5.9 and if $U \subset W$ is a generic 3-dimensional subspace containing K_p then letting $\Sigma + \Gamma = [H_2' \cap H_3' \cap H_4']$ where $\langle H_2', H_3', H_4' \rangle = \mathbb{P}(U)$, we have $p \in \Gamma$ (Σ is reduced from the proof of the first statement of the Claim). Now let p_0 be as is the statement of the Claim; then $\dim K_{p_0} \geq 2$; it follows that the subset of $Gr(3, W) = \mathbb{P}(W^\vee)$ defined by

$$\{U \mid \delta_{p_0}(U) \neq im(\delta_{p_0})\}$$

is a linear subspace of dimension at most 1. Since the set of $U \in Gr(3, W)$ containing K_p is a 2-dimensional linear subspace there exists $U_0 \in Gr(3, W)$ containing K_p which does not belong to the set of the above equation and such that the corresponding Σ is reduced. Let $\langle H_2', H_3', H_4' \rangle = \mathbb{P}(U_0)$. If $\langle H_1', H_2', H_3', H_4' \rangle = \Theta$ then $mult_p(H_1' \cdot \Gamma)) \geq 1$ and $mult_{p_0}(H_1' \cdot \Gamma)) \geq 2$ hence the Claim follows from (5.1). □

Thus we see that either:
  (1) the divisor Γ' is Cartier and define a base-point-free linear system on D, or
  (2) the divisor Γ' is Cartier and $|\Gamma'|$ has only isolated base points that are outside Σ', or
  (3) we have $d = 7$, there is a unique point $P_0 \in \Sigma'$ such that $P_0 \in \Gamma'$ for each Γ' (for a generic choice of H_3').

• Assume we are in case (1). Then by [DH, Thm. 2.1] the surface D has only isolated singularities at singular points of Σ'. Let $(\tilde{D}, \Sigma^\circ)$ be a minimal resolution of (D, Σ), $\tilde{\Gamma}$ the pull-back of Γ' on $\tilde{D}$, and Σ° the strict transform of Σ'.

We claim that Σ' is irreducible. Indeed, if Σ' has two components Σ_1 and Σ_2 (denote by Σ_1°, Σ_2° the corresponding components of Σ°) then from Claim 9.1 we have $\Sigma_1^\circ \cdot \tilde{\Gamma} \geq 1$ and $\Sigma_2^\circ \cdot \tilde{\Gamma} \geq 1$ on D. This contradicts (5.1).

By (5.1) we have $\Sigma' \cdot \Gamma' = 2$ (resp. 3) if $d = 8$ (resp. 7) (observe that if $p \in \Sigma'$ is smooth then $mult_p(\Sigma' \cdot \Gamma') = mult_p(H_3 \cdot \Gamma')$, where $H_3 \in |H|$ is generic). Thus, the image of Σ° by $|\tilde{\Gamma}|$ is a smooth conic or a line in $X_D' \subset \mathbb{P}^3$ (resp. a rational normal curve in $\mathbb{P}^3$, a smooth elliptic curve, a line, or a singular cubic curve in $\mathbb{P}^2$).

Proposition 9.2. *The morphism given by $|\Gamma'|$ is the normalization of X_D'.*

Proof. We have to prove that φ does not contract divisors on X to surfaces. Suppose that an irreducible divisor $S \subset X$ is contracted to a surface. Let $\pi : \tilde{X} \to X$ be the blow up of $\Sigma' \subset X$. Then as in the proof of Lemma 8.1 the image $I = \varphi_{|\pi^*(H)-E|}(E) \subset \mathbb{P}^5$ is covered by planes.

First, assume that $\Sigma' \subset D$ maps by $|\Gamma'|$ to a line. It follows that the generic 3-dimensional linear section of I is a contained in a line. Thus, I is contained in a 3-dimensional linear space in $\mathbb{P}^5$. We conclude by Lemma 5.2. If the image of Σ° is not a line then $|\tilde{\Gamma}|$ is generically $1:1$ on Σ°.

Let us now consider the case where there is a point $P \in \Sigma'$ such that for a generic curve $C \subset S$ contracted by φ we have $P \in C$. Then the image of C is contained in the plane in $\mathbb{P}^5$ being the image of the fiber of the exceptional divisor E under P. We can conclude as in Lemma 5.2.

Let us consider the remaining cases. Suppose that $C \subset S$ is a curve on D contracted by φ (denote by C° its strict transform on $\tilde{D}$). We can assume that C intersects Σ' in smooth points on Σ' (thus smooth on D). The divisors $\tilde{\Gamma}|_{\Sigma^\circ}$ gives a linear system Λ on Σ°. We have $C^\circ \cdot \Sigma^\circ = C \cdot H \geq 2$, thus if $P \in \Sigma^\circ \cap C^\circ$ and $P + A \in \Lambda$, where A is an effective divisor on Σ°, then $\text{supp } A \cap C^\circ \cap \Sigma^\circ$ is non-empty. Now, observe that the only linear systems of degree 2 with this property are one-dimensional. Moreover, the images of such linear systems of degree 3 that are not lines are singular (so they are plane cubics).

Let us assume that the image by $|\Gamma'|$ of $\Sigma' \subset D$ is a plane cubic. It follows that the image of E in $\mathbb{P}^5$ is contained in a hyperplane L. Since $S \in |kH|$ with $k \geq 1$ and the image of S is contained in L we obtain $k = 1$ and $C + \Sigma' \in |H||_D$ (because S cannot be a proper component of the pre-image of L). Thus $C^\circ \cdot \Sigma^\circ = 3$, so the

linear system Λ has the property; if $P \in \Sigma^\circ \cap C^\circ$ and $P + A \in \Lambda$, where A is an effective divisor on Σ°, then $P + A = C^\circ \cdot \Sigma^\circ$. It follows that the image of Σ' is a line, a contradiction. $\square$

We deduce that the conductor of the normalization of X'_D defines a locally CM subscheme $C \subset \mathbb{P}^3$ such that $2 \deg C = \Gamma'((d-6)\Gamma' - 2\Sigma')$. We obtain a contradiction if $d = 7$. So assume that $d = 8$, thus $\deg(C) = 6$. We shall compute $h^0(\mathcal{I}_C(n+4)) = h^0(K_D + n\Gamma')$ for $n = -2, -1, 0, 1$. We have $h^0(2\Gamma' + 2\Sigma') = h^0(D, \mathcal{O}_D(2H)) = h^0(K_D)$, thus from Theorem 1.1 (6) we obtain $h^0(\mathcal{I}_C(4)) = 11$. Since $h^0(\mathcal{I}_C(1)) = 0$ and $h^0(2\Sigma') \geq 1$, we have $h^0(\mathcal{I}_C(2)) = 1$. It follows that $h^0(\mathcal{I}_C(3)) \geq 4$. In this case $\chi(K_D + n\Gamma') = 4n^2 + 10n + 12$ and

n	$h^0(K_D + n\Gamma')$	$h^1(K_D + n\Gamma')$	$h^2(K_D + n\Gamma')$	$\chi(K_D + n\Gamma')$
0	11	0	1	12
-1	$4 \leq$	x	4	6
-2	1	y	$7+y$	8
-3	0	z	$18+z$	18
-4	0	t	$36+t$	36

where as before $4 \geq y \geq 3$, $z \geq 2$, and $x \geq 2$. Let $B \subset \mathbb{P}^3$ be a degree 4 curve 2×5 linked to C.

n	$h^0(\mathcal{I}_B(n))$	$h^1(\mathcal{I}_B(n))$	$h_B(n)$
3	$t+5$	t	0
2	$z-1$	z	$x-2$
1	$y-3$	y	a
0	0	x	b

If $x > 2$ then $h_C(2) \leq -1$ contradiction, thus $x = 2$, $a = 1$, and $b = 3$. It follows that B is not extremal (see [S3]) and that $p_a(B) \geq -2$. We have the following inequalities from [N] (see [S3, Thm. 4.4]): $y \leq 3$, $z \leq 2$, $t \leq 1$. Thus we have two possibilities $(y, z, t) = (3, 2, 0)$ or $(3, 2, 1)$. It follows that B is contained in a quadric and $(h_C(0), h_C(1), h_C(2))$ is equal to $(1, 4, 1)$ or $(2, 2, 2)$. We infer from [S2, Cor. 4.4] that B is minimal in its biliaison class, and C can be bilinked down on the quadric to C_0 a minimal curve of degree 2 (we use (2.6)). We obtain a contradiction with [Mi, Ex. 1.5.10] where all the possible deficiency modules of non reduced curves of degree 2 are described.

• Assume we are in case (2). Then the 0-dimensional components of the base locus of $|H|$ have length ≤ 2 and Σ' is reduced and irreducible (from (5.1)). Thus from Lemma 5.6 the surface D is smooth outside Σ', so has only isolated singularities (from [DH, Thm. 2.1]). We have also that $\Gamma' \cdot \Sigma' \leq 2$. Thus the image of Σ' is a smooth conic or a line. We prove as in Proposition 9.2 that φ does not contract curves on D. Thus $\rho_D \circ \beta_D$ gives the normalization of X'_D and the conductor of the normalization of X'_D defines a locally CM subscheme $C \subset \mathbb{P}^3$ such that $2 \deg C = \Gamma''((d-6)\Gamma'' - 2\Sigma' - R)$, where R is an effective divisor supported on

the exceptional lines on $\overline{D}$ and Γ'' (resp. Σ'') the strict transform of Γ' (resp. Σ') on $\overline{D}$. Now, if $d = 7$ we obtain a contradiction with $\deg(C) \geq 1$.

Assume that $d = 8$. Then $\Gamma' \cdot \Sigma' = 1$ and Γ' has exactly one isolated simple base point P_0, denote by E the exceptional divisor of the blow-up at P_0. From the adjunction formula we infer

$$2g(\Gamma'') - 2 = \Gamma''(\Gamma'' + K_{\overline{D}}) = \Gamma''(3\Gamma'' + 2\Sigma'' + 3E) = 29,$$

a contradiction since the genus $g(\Gamma'')$ is an integer.

- Assume we are in case (3).

Lemma 9.3. *The intersection D of two generic divisors $H_1', H_2' \in |H|$ is smooth at P_0.*

Proof. Arguing as in the proof of [O, Prop. 5.4(2)] we infer that the generic Γ' is smooth at P_0; moreover, the tangent direction of Γ' is not contained in the tangent space $T_{P_0}\Sigma'$. If H_1 is singular at P_0, then the multiplicity of the intersection of three generic divisors from Θ at P_0 is ≥ 8. It follows that Σ' is singular and $T_{P_0}\Sigma'$ has dimension ≥ 3. Thus the tangent space $T_{P_0}\Sigma'$ cannot intersect transversally $T_{P_0}\Gamma'$, a contradiction. Repeating the above arguments for H_1' instead of X we end the proof. $\square$

It follows that Σ' and Γ' are Cartier divisors. Denote by E the exceptional divisor and by Σ'' the strict transform of Σ'. From (5.1) we infer that P_0 is a simple base point of $|\Gamma'|$, i.e. the strict transform Γ'' of Γ' on the blowing-up D' of D at P_0 is base-point-free. It follows that we can resolve the indeterminacy of $|H|$ by blowing-up Σ' and then the fiber over P_0 of the obtained exceptional divisor. Denote by $\overline{X}$ the obtained threefold and by $E_1 \subset X$, $E_2 \subset X$ the resulting exceptional divisors.

We claim that the morphism $\overline{\varphi} \colon \overline{X} \to X'$ induced from φ maps E_1 and E_2 into two 3-dimensional linear subspaces of $\mathbb{P}^5$. Indeed, it is enough to observe that E and Σ'' maps into generic hyperplane sections of E_1 and E_2. Now, the image of E is a line since Γ' is smooth at P_0 and P_0 is a simple base point. The image of Σ' is also a line since $\Sigma'' \cdot \Gamma'' = 1$, the claim follows.

So, we can use Lemma 5.2 to prove the following:

Lemma 9.4. *The morphism given by $|\Gamma''|$ does not contract curves.*

It follows that, $|\Gamma''|$ is the normalization of X_D', the given codimension 2 linear section of $X' \subset \mathbb{P}^5$. We infer that the conductor this normalization defines an CM subscheme $C \subset \mathbb{P}^3$ such that $2 \deg(C) = \Gamma''(\Gamma'' - 2\Sigma'' - sE)$ for some $s \geq 5$. This is a contradiction since $\Gamma'' \cdot E = 1$, $\Gamma'' \cdot \Sigma'' = 1$, and $\deg(C) > 0$.

9.2. **Suppose next that the base locus B is 0-dimensional.** As before, we denote by D the intersection of two generic elements of $|H|$ and set $H|_D = \Gamma'$. From Lemma 5.6 we infer that D is smooth. The new cases are when $\operatorname{supp} B$ is one point P, where B is the base locus of $|H|$.

- Suppose first that $d = 8$. Denote by $(\overline{D}, \overline{\Gamma})$ the Hironaka model of (D, Γ'). Consider the Stein factorization $\overline{D} \to Y_D \to X_D'$ of the morphism given by $|\overline{\Gamma}|$.

Assume moreover that the generic Γ' is smooth at P. Then as in the case $d = 9$ and $\dim B = 0$, we see that the Hironaka model $\overline{D}$ is obtained by four blowings-up at each step of the unique fixed point of the linear system $|\overline{\Gamma}|$ which is the strict transform of the linear system $|\Gamma'|$. We have however five possible configurations of the resulting exceptional curve (depending on the positions of fixed points on

exceptional divisors). In this case the morphism $\rho : \overline{D} \to Y_D$ is birational and contracts all the exceptional divisors except the last one. From [Ar, Thm. 3] we infer that the singularities of Y_D are Du Val or rational triple points (see [Ar, p. 135]). We deduce as before that ρ_D does not contracts curves. Since a surface with rational singularities is $\mathbb{Q}$-factorial and Cohen–Macaulay, we can argue as before and conclude that the ideal of the conductor needs at least 11 generators.
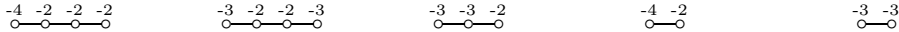
If Γ' is singular at P then it has multiplicity 2 there. Then $\overline{D}$ is the blowing-up of D at P, denote by E the exceptional divisor. The strict transform $\overline{\Gamma}$ is base-point-free, because $\overline{\Gamma}$ is semi-ample and $\overline{\Gamma}^2 = 8$.

Lemma 9.5. *The morphism ρ_D does not contract curves.*

Proof. Suppose that the curve $C \subset D$ is contracted by φ, then we have $P \in C$. Thus the image of C is contained in the 3-dimensional component of the image of the exceptional locus of the Hironaka model of (X, H) (obtained by blowing-up a point then a line in the exceptional divisor, we are interested in the second one). Now, since $\overline{\Gamma}|_E$ has degree 2 the above components maps to a quadric $Q \subset \mathbb{P}^5$ or to a 3-dimensional linear subspace. If the image is linear we can apply the Lemma 5.2. Let us assume that it is a quadric. Then the quadric Q is contained in a hyperplane M. To end the proof of the Lemma it is enough to show (by the proof of Lemma 5.2) that Q is a proper component of $X' \cap M$. Suppose the contrary, then each curve C contracted by φ is an element of $|\Gamma'|$. Now, the linear system $|\overline{\Gamma}||_E$ is 2-dimensional and $\overline{C} \cdot \overline{\Gamma} = 0$. If $\overline{C} \cdot E = m$, then $m\overline{\Gamma} \cdot E = \overline{\Gamma}(\overline{C} + mE) \geq 8$ so $m \geq 4$. Thus $|\overline{\Gamma}||_E$ is 0-dimensional, a contradiction. $\square$

As before, the conductor of the normalization of X'_D defines a locally CM subscheme $C \subset \mathbb{P}^3$ such that $\deg(C) = 2$. Now, or C is reduced then it is an aCM plane curve or a double line with Hartshorne–Roe module described in [Mi, Ex. 1.5.10]. We find as before $h^1(\mathcal{I}_C(3)) = h^1(K_{\overline{D}} - \overline{\Gamma}) = 6$ and $h^1(\mathcal{I}_C(n)) = 0$ for $n > 3$ a contradiction.

• Suppose now that $d = 7$. Then if Γ' is smooth at P the Hironaka model $\overline{D}$ is obtained as before by five successive blow-ups. With the notation as above, the possible singularities on Y_D are Du Val singularities, rational triple points, cyclic singularities of type $\frac{1}{3}(1,1)$, $\frac{1}{4}(1,1)$, $\frac{1}{5}(1,1)$, and singularities whose minimal resolutions have exceptional curves with the following configurations:

In the figure "o" denotes a nonsingular rational curve with self-intersection equal to the number above it. In each case the fundamental cycle is equal to the reduced curve and the arithmetic genus is 0. Thus by [Ar, Thm. 3.5] the singularities on Y_D are rational, and we conclude as before.

If Γ' is singular at P then it has multiplicity 2 there. The strict transform of Γ' on the blow-up of D at P has self-intersection 8, thus it has a base point on the exceptional divisor. Blowing-up this point we obtain the Hironaka model $(\overline{D}, \overline{\Gamma})$ of (D, Γ').

We claim that the morphism given by $|\overline{\Gamma}|$ does not contract curves. Since $|\overline{\Gamma}|$ maps the exceptional divisors on $\overline{D}$ into two lines, we can argue as in Lemma 9.5.

We infer that the subscheme C given by the conductor is locally CM moreover $\deg(C) < 0$ a contradiction.

Remark 9.6. It was observed by the referee that the case $d = 7$ and $\dim B = 0$ can be dealt with by comparing the geometric genus of a generic $\Gamma' = H_2 \cap H_3 \cap H_4$ and the corresponding (birational) plane septic curve $C = L_1 \cap L_2 \cap L_3 \cap X'$; on one hand $p_g(\Gamma') \geq 18$ on the other hand $p_g(C) \leq 15$.

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